INDIAN STATISTICAL INSTITUTE
Mid-Semestral Exam
Algebra-I
2018-2019

Total marks: 100
Time: 3 hours

Answer all questions.

1. Let $\mathbb{Z}_n$ and $\mathbb{Z}_m$ be two cyclic groups of order n and m respectively. State and prove a necessary and sufficient condition involving n and m for $\mathbb{Z}_n \times \mathbb{Z}_m$ to be cyclic. (12)

2. Prove that if a finite group G is cyclic of order n, then for every positive integer d dividing n, there exists a unique subgroup of order d in G. (14)

3. (a) If a finite group G acts on a finite set S, show that each orbit has cardinality a divisor of the order of G.
(b) Let G be a group of oder p^n for some prime p acting on a finite set S whose cardinality is not a multiple of p. Show that there exists a $x_0 \in S$ such that $g.x_0 = x_0$ for all $g \in G$. (8+8)

4. (a) Find the distinct conjugacy classes of A_4.
(b) Determine the class equation for A_4. (8+4)

5. Let G be a finite group and p be a prime dividing the order of G. Prove that there exists an element of order p in G. (You may treat the abelian and non-abelian cases separately.) (16)

6. Give examples of the following and justify your answers.
(a) Two elements $g, h \in G$ such that g and h are of finite order, but gh is of infinite order.
(b) An infinite group G, all of whose elements have finite order.
(c) A group G and subgroups H and K such that $H \trianglelefteq K$, $K \trianglelefteq G$ but $H \ntrianglelefteq G$.
(d) A group G with normal subgroups H and K, such that $H \cong K$ but $G/H \ncong G/K$.
(e) A group G and a non-trivial normal subgroup N such that $G/N \cong G$. (5 × 6)

——— *End* ———